MATH 1A - MIDTERM 1 - SOLUTIONS

PEYAM RYAN TABRIZIAN

1. *(10 points)* Find the domain of $f(x) = \ln(x) + \sqrt{1-x^2}$

We want:

(i) $\boxed{x > 0}$ (because the number under the $\ln$ has to be positive)

(ii) $1 - x^2 \geq 0$ (the number under the $\sqrt{}$ has to be nonnegative), which is the same as $x^2 \leq 1$, which is the same as $\boxed{-1 \leq x \leq 1}$

Combining those two facts (draw a picture if necessary), we get that the domain of f is: $\boxed{0 < x \leq 1}$, that is, $\boxed{(0, 1]}$

Date: Friday, July 1st, 2011.

2. *(10 points, 5 points each)* In the following problem, you do **not** have to graph the resulting functions. **BE BRIEF!**

Note: You may use the words 'Shift up/down/left/right', 'Stretch/Compress horizontally/vertically by a factor of $\cdots$' and 'Flip about the x/y - axis'.

(a) Explain in words how to obtain the graph of $y = 2 - x^2$ from the graph of $y = x^2$

1) Flip the graph of $y = x^2$ about the x-axis

2) Shift the resulting graph up 2 units.

(b) Explain in words how to obtain the graph of $y = \cos(2x + 3)$ from the graph of $y = \cos(x)$

1) Shift the graph of $y = \cos(x)$ to the left 3 units

2) Compress the resulting graph horizontally by a factor of 2

3. *(10 points)* Find $f^{-1}(x)$, where $f(x) = 1 + e^{x^3}$

Note: Make sure to write your final answer in terms of x.

1) Let $y = 1 + e^{x^3}$

2)

$$\begin{aligned} y &= 1 + e^{x^3} \\ y - 1 &= e^{x^3} \\ \ln(y-1) &= \ln(e^{x^3}) \\ \ln(y-1) &= x^3 \\ x^3 &= \ln(y-1) \\ x &= \sqrt[3]{\ln(y-1)} \end{aligned}$$

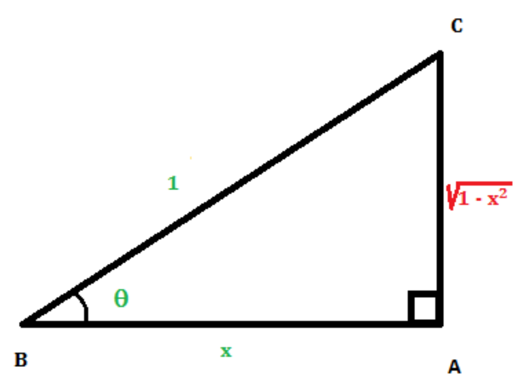
3) Hence $\boxed{f^{-1}(x) = \sqrt[3]{\ln(x-1)}}$

4. *(15 points)* Evaluate $\tan(\cos^{-1}(x))$

Note: Show your steps. You are not just graded on the correct answer, but also on the way you write up your answer.

Let $\theta = \cos^{-1}(x)$, then $\cos(\theta) = x$.

1A/Handouts/Examtriangle.png



Then:

$$\tan(\cos^{-1}(x)) = \tan(\theta) = \frac{AC}{AB} \overset{PYTH}{=} \frac{\sqrt{1-x^2}}{x}$$

5. *(40 points, 5 points each)* Evaluate the following limits (or say 'it does not exist'). **Briefly show your work!, and do NOT use l'Hopital's rule** :

(a)

$$\lim_{x\to\infty} \frac{1}{2x-1} = \frac{1}{\infty} = 0$$

(b)

$$\lim_{x\to 3^+} \frac{e^x}{x-3} = \frac{e^3}{0^+} = \infty$$

(c)

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt{x+3}-2}{x-1} &= \lim_{x\to 1} \frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)} \\ &= \lim_{x\to 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} \\ &= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)} \\ &= \lim_{x\to 1} \frac{1}{\sqrt{x+3}+2} \\ &= \frac{1}{\sqrt{4}+2} \\ &= \frac{1}{4}\end{aligned}$$

(d)

$$\lim_{x\to\infty} \frac{x^2+2x+3}{x^2-2x+3} = \lim_{x\to\infty} \frac{x^2(1+\frac{2}{x}+\frac{3}{x^2})}{x^2(1-\frac{2}{x}+\frac{3}{x^2})} = \lim_{x\to\infty} \frac{1+\frac{2}{x}+\frac{3}{x^2}}{1-\frac{2}{x}+\frac{3}{x^2}} = \frac{1}{1} = 1$$

(e)

$$\lim_{x\to 2} \frac{x^2-4x+4}{x^2-2x+3} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x-2}{x-1} = 0$$

(f)

$$\begin{aligned}\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x} &= \lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x}\\ &= \lim_{x\to-\infty}\frac{|x|\sqrt{1+\frac{1}{x^2}}}{x}\\ &= \lim_{x\to-\infty}\frac{-x\sqrt{1+\frac{1}{x^2}}}{x}\\ &= \lim_{x\to-\infty}-\sqrt{1+\frac{1}{x^2}}\\ &= -1\end{aligned}$$

(g) $\lim_{x\to 0}\frac{|x|}{x}$

$$\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

$$\lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

Since the left-hand-side and the right-hand-side limits are unequal, the limit **does not exist**

(h) $\lim_{x\to 0} x^4\cos(\frac{4}{x})$

$-1 \leq \cos\left(\frac{4}{x}\right) \leq 1$, so $-x^4 \leq x^4\cos(\frac{4}{x}) \leq x^4$.

But since $\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0$, by **the squeeze theorem**, $\lim_{x\to 0} x^4\cos(\frac{4}{x}) = 0$.

6. *(15 points)* Show that the equation $x^3 + x - 1 = 0$ has at least one solution. **Show your work: You will be graded not only on the correct answer, but also on the way you write up your answer**

Let $f(x) = x^3 + x - 1$.

Then $f(0) = -1 < 0$, and $f(1) = 1 > 0$, and since f is **continuous** (it is a polynomial), by **the intermediate value theorem**, $f(x)$ has at least one zero on $[0, 1]$, hence $x^3 + x - 1 = 0$ has at least one solution.

Bonus 1 *(5 points)* Suppose you are taking an elevator up from the 1^{st} floor to the 3^{rd} floor, and suppose that at the same time, your friend is taking a similar elevator down from the 3^{rd} floor to the 1^{st} floor. Assuming that the motion of the elevators is continuous, and that both of you get out of your elevators after 1 minute, show that there is one time where you two are on the same level. (so if the elevators were transparent, you could wave at your friend)

Hint: Let $g(t)$ be the height of your elevator at time t, and let $h(t)$ be the height of your friend's elevator at time t. Consider the function $f(t) = g(t) - h(t)$

This is very similar to the previous question:

Let f, g, h be defined as in the hint.

Then $f(0) = g(0) - h(0) = 1 - 3 = -2 < 0$, and $f(1) = g(1) - h(1) = 3 - 1 = 2 > 0$. And since f is continuous (assuming that the motion of the two elevators is continuous), by the Intermediate Value Theorem, there is one time c where $f(c) = 0$.

At that time c, $f(c) = g(c) - h(c) = 0$, so $g(c) = h(c)$, which means that at time c, the two elevators have the same height.

Bonus 2 *(5 points)* Let f be any function.

(a) Show that the function $g(x) = \frac{f(x)+f(-x)}{2}$ is always even.

$$g(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = g(x)$$

So g is even.

(b) Show that the function $h(x) = \frac{f(x)-f(-x)}{2}$ is always odd.

$$h(-x) = \frac{f(-x) - f(x)}{2} = \frac{-(f(x) - f(-x))}{2} = -h(x)$$

So h is odd.

(c) Using (a) and (b), show that any function f can be written as a sum of an even function and an odd function. (this is called the **even/odd decomposition of** f, and in fact it is unique)

Let g and h be defined as above. Then:

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{2f(x)}{2} = f(x)$$

So $f(x) = g(x) + h(x)$, which says precisely that f is the sum of an even function (g) and an odd function (h).

(d) If $f(x) = e^x$, calculate g and h as in (a). Do you happen to know the names of those two functions?

$g(x) = \frac{e^x+e^{-x}}{2} = \cosh(x)$ and $h(x) = \frac{e^x-e^{-x}}{2} = \sinh(x)$. So, in other words, $\cosh(x)$ is the even part of e^x and $\sinh(x)$ is the odd part of e^x. How cool is that?